Signature Changing Spacetimes and WKB Approximations in General Relativity

Sean A. Q. Young

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Abstract

Some properties of spacetimes that change signature from Riemannian (positive definite metric) to Lorentzian (metric has a single negative eigenvalue) are investigated. Specifically, the form of geodesics and solutions to the Klein-Gordon equation are calculated. Geodesics behave as expected since they are equally well defined for Lorentzian and Riemannian manifolds, though null geodesics cease to have meaning in the Riemannian region. Solutions to the Klein-Gordon equation exhibit oscillatory behavior in the Lorentzian region and exponential behavior in the Riemannian region. In an effort to further interpret these results, approximate wave solutions are found for a generic spacetime using WKB approximations in the large momentum limit. This approximation encompasses traditional, non-degenerate spacetimes as well as those that change signature. This solution is shown to break down near regions where the metric becomes degenerate, except in the $1+1$ dimensional case. Further, these solutions can define a vector field with the gradient of their phase. The integral curves of the resulting field are shown to be geodesics, parametrized by an affine parameter.

Signature Changing Spacetimes and WKB Approximations in General Relativity

Presented by
Sean Alden Quigg Young
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Approved by

Richard A. Matzner	Date
Greg O. Sitz	Date

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Department of Physics

Sean A. Q. Young　　Date

Richard A. Matzner　　Date

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Acknowledgments

I am grateful to Luis R. Suazo for his assistance in the entire process of researching, compiling, and composing this thesis, as well as the many hours we spent working out the minute details. Before this project, I had only a rudimentary understanding of general relativity and differential geometry. I still have barely scratched the surface of the subject, but I am far more comfortable in curved spacetime than before. I hope to continue learning about the subject throughout my life, even after leaving the field.

I thank Dr. Richard Matzner for allowing me to work in the Center for Relativity for two years. Though the scope of the work being done throughout the Center tends to fly far above the training of a lowly undergraduate, I grew enormously as a physicist in my time here.

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Lastly, I am eternally grateful to my friends and family. Thank you Mom and Dad for showing me that knowledge is its own reward, for introducing me to works of science fiction and popular science that have influenced me academically for my whole life, and for supporting me in my pursuit of my childhood dreams. Thank you Elizabeth for enduring both my agony when research and schoolwork demands seemed insurmountable and my great joy when everything began to fall into place. Our evening calls helped clear my head more times than you know. I love you all and I can't begin to thank you enough for the support you have given me.

> "Passion is what gets you through the hardest times that might otherwise make strong men weak, or make you give up."
>
> \- Neil deGrasse Tyson

Chapter 1

Introduction

The purpose of this Chapter is to briefly motivate the work that follows, as well as giving a short introduction to General Relativity (GR) for readers interested in the results, but who are unfamiliar with the machinery of differential geometry. Section 1.1 describes the formalism required for subsequent sections and closely follows the description from Carroll [1]. Section 1.2 discusses the work by Stephen Hawking and James Hartle [2] that led to a portion of this research, as well as the impetus for the later work in WKB solutions in Chapter 3. Section 1.3 gives some background into asymptotic analysis - specifically the WKB approximation - and the limitations to this method.

1.1 Background in General Relativity

1.1.1 Essentials

Einstein's General Theory of Relativity ascribes gravitational effects to the curvature of spacetime. In the Newtonian regime, a gravitational field due to a mass can be described as a force field that permeates space. Objects then propagate according to Newton's second law. Relativity replaces this vector field with the notion that massive objects warp space and time around them. This warping distorts straight lines into more general geodesic curves that bend toward masses. Free particles now move along these curves. Because spacetime itself is no longer flat, this new description requires the language of differential geometry [1].

Formally, spacetime is a differential manifold equipped with a metric that forms the background for other physical phenomena. At any event in the manifold (a generalization of point to include position and time) the manifold appears flat and space appears to be the classical Euclidean space of classical geometry. Globally, however, the spacetime can be drastically different.

A fundamental quantity that describes spacetime is the metric tensor, with components $g_{\mu\nu}$. The metric defines the line element of the spacetime as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

where the Einstein summation convention is used [2]. The inverse metric $g^{\mu\nu}$ is defined such that $g^{\mu\lambda} g_{\nu\lambda} = \delta^\mu_\nu$, where δ^μ_ν is the Kronecker delta.

$$\delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases} \tag{1.2}$$

Minkowski (flat) spacetime is given the special metric symbol of $\eta_{\mu\nu}$ and the line element (for a $1+3$ dimensional spacetime) in units such that $c = \hbar = 1$ is given by

[1]Throughout, component notation is used. Thus rather than writing vectors and dual vectors as V or ω I denote them by their components V^μ or ω_μ.

[2]This means that whenever an index appears as both a subscript and a superscript, it is summed over. For example, $a_\mu b^\mu$ really means $\sum_{\mu=0}^{d} a_\mu b^\mu = a_0 b^0 + a_1 b^1 + ... + a_d b^d$.

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\
&= -dt^2 + dx^2 + dy^2 + dz^2
\end{aligned} \tag{1.3}
$$

Other useful quantities in differential geometry can be constructed from the metric and are at the core of GR. Another piece of the puzzle are the connection coefficients $\Gamma^{\rho}_{\mu\nu}$. A connection provides a method for linking separate tangent spaces in the manifold, facilitating comparison and use of calculus with covariant derivatives. For scalar functions, vectors, and dual vectors, respectively, covariant derivatives take the form:

$$
\nabla_\mu \phi = \partial_\mu \phi \tag{1.4}
$$

$$
\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\rho} V^\rho \tag{1.5}
$$

$$
\nabla_\mu \omega_\nu = \partial_\mu V_\nu - \Gamma^{\rho}_{\mu\nu} \omega_\rho \tag{1.6}
$$

Generalizing this to tensors is straightforward; simply add connection terms for each additional index. In principle there are an infinite number of connections that could be used. In practice, imposing the restriction that the connection be torsion-free (symmetric in lower indices) and that the covariant derivative of the metric vanishes, the connection has a very specific form: the Levi-Civita Connection. In components, they are known as the Christoffel coefficients, given by

$$
\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right). \tag{1.7}
$$

Now it is possible to define a notion of parallel transport, geodesics, and finally curvature.

1.1.2 Curvature and Matter

The connection tells us how to transport geometric information from one event in spacetime to another. In particular, it relates two disparate tangent spaces through parallel transport. This is a process by which vectors moved along a curve remain (locally) parallel. This allows you to compare vectors at different regions of the manifold.

Geodesics are the generalization of straight-line motion in Euclidean space to curved spacetime. They have the interpretation that they are the paths on which free-particles in a spacetime travel. These curves are constructed such that they parallel transport their own tangent vectors. For an arbitrary parametrization of the curve, geodesics $x^\mu(\lambda)$ satisfy the differential equation

$$
\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = f(\lambda) \frac{dx^\mu}{d\lambda} \tag{1.8}
$$

where $f(\lambda)$ is a function that depends on the parametrization. This is a direct analogue to Newton's second law for free particles. Now, however, there are terms that depend on the metric and on the parameter. If $f(\lambda)$ vanishes, λ is called affine parameter and is related linearly to the proper time of a particle moving along the geodesic. If λ is not affine, then $f(\lambda)$ can be expressed as a deviation of this parameter from some affine parameter σ

$$
f(\lambda) = -\frac{d^2 \lambda}{d\sigma^2} \left(\frac{d\lambda}{d\sigma} \right)^{-2} \tag{1.9}
$$

The connection term vanishes in Euclidean space for Cartesian coordinates. Thus, using an affine parameter in flat spacetime recovers Newton's second law.

The curvature of the manifold itself is not trivial to describe. Unlike two-dimensional surfaces, more than one component is required to completely specify the manifold. For a $1+d$ manifold, $d(d+2)(d+1)^2/12$ independent components need specification. The Riemann curvature tensor encodes this information and is defined as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{1.10}$$

This tensor has a high degree of symmetry. Swapping different indices yields

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu} = R_{\mu\nu\rho\sigma} \tag{1.11}$$

while other symmetries exist by taking the covariant derivative or by looking only at the antisymmetric part of this tensor[3]. If you contract this tensor along the first and third indices, you get another useful tensor: the Ricci tensor.

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} \tag{1.12}$$

Contracting again, yields get the Ricci scalar.

$$R = g^{\mu\nu} R_{\mu\nu} \tag{1.13}$$

Now we have all the tools needed to construct the dynamical equations of GR. The Einstein tensor is formed from the Ricci tensor and scalar and has the following form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.14}$$

The connection of curvature to matter content and dynamics is a complicated one. They are related through the Einstein field equations,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1.15}$$

where G_N is Newton's constant and $T_{\mu\nu}$ is the stress-energy tensor describing the mass and energy within the spacetime. Using these equations, one can determine the spacetime curvature given the matter content of the universe and vice versa. Coupled with the geodesic equation, you can also determine how test particles and light move through the manifold.

1.1.3 Klein-Gordon Equation

Waves are complicated somewhat by existing in curved rather than flat spacetime. For instance, an expanding universe or intense gravity wells create the well known gravitational redshift of light - the increasing of wavelength or, equivalently, the decrease in energy of light waves as they climb out of a planet's gravity. The Klein-Gordon equation will be used in the following sections as our wave equation of interest. It is defined as

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = m^2 \phi \tag{1.16}$$

where ϕ is the wave solution and m is the mass of the field. Note that for Minkowski spacetime and $m = 0$ this reduces to the usual wave equation. While it has a quantum mechanical interpretation as a relativistic Schrödinger equation for massive spin-0 particles, it is also the classical equation of motion for a scalar field with a harmonic oscillator potential. While this expression can be expanded in terms of connection coefficients, an identity allows the Klein-Gordon equation to be written in a more useful form. The covariant derivative of a vector, contracted along its indices can be written as

[3]These are the Bianchi identities and are not used here. They are $R_{\rho[\sigma\mu\nu]} = 0$ and $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$.

$$
\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \\
&= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}} \partial_\lambda \sqrt{-g}\, V^\lambda \\
&= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, V^\mu\right)
\end{aligned}
$$

(1.16) can be written as $\nabla_\mu \left(g^{\mu\nu} \nabla_\nu \phi\right) = m^2$ since $\nabla_\mu g^{\rho\sigma} = 0$. Thus, letting $V^\mu = g^{\mu\nu} \nabla_\nu \phi$,

$$
\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi\right) = m^2 \phi. \tag{1.17}
$$

In this form certain properties become clear. If the metric components $g_{\mu\nu}$ are products of functions of a single coordinate, then the resulting differential equation becomes separable and solutions are far easier to find. For instance, in a 1+1 dimensional manifold, suppose the metric took the form

$$
ds^2 = -a(t) dt^2 + b(t) h(x) dx^2 \tag{1.18}
$$

then $g = -a(t)b(t)h(x)$ and the Klein Gordon equation takes the form

$$
\begin{aligned}
\frac{1}{\sqrt{abh}} \partial_\mu \left(\sqrt{abh} g^{\mu\nu} \partial_\nu \phi\right) &= m^2 \phi \\
-\frac{1}{a} \left[\frac{\partial_0 \sqrt{ab}}{\sqrt{ab}} - \frac{\partial_0 a}{a} + \partial_0\right] \partial_0 \phi + \frac{1}{bh} - \left[\frac{\partial_1 \sqrt{h}}{\sqrt{h}} \frac{\partial_1 h}{bh} + \partial_1\right] \partial_1 \phi &= m^2 \phi
\end{aligned}
$$

It is clear that if the wavefunction ϕ is separable, that the above equation can be split into the sum of two functions of a single variable. This decomposition is discussed further in Chapter 3.
In $1+1$ Minkowski (flat) spacetime, (1.17) becomes

$$
\left[-\partial_t^2 + \partial_x^2 - m^2\right] \phi = 0
$$

which has as a solution

$$
\phi = A \cos\left(kx - \omega t + \eta_+\right) + B \cos\left(kx + \omega t + \eta_-\right) \tag{1.19}
$$

where A, B, $\eta_\pm$ and k are constants and $\omega = \sqrt{k^2 + m^2}$. Solutions to the Klein-Gordon equation are evidently monochromatic plane waves or linear combinations thereof. Considering only the first term in the above equation (let $B = 0$), a few observations motivate the discussion that follows. The equation $\theta = kx - \omega t + \eta_+$, where θ is a constant defines lines in the xt plane along which ϕ takes a constant value; these are the wavefronts. The gradient of the argument of the function gives how these wavefronts vary in time and space. Denoting this gradient $V^\mu = g^{\mu\nu} \partial_\nu \theta$, its components are

$$
V^0 = \sqrt{k^2 + m^2} \tag{1.20}
$$

$$
V^1 = k \tag{1.21}
$$

The norm of this gradient is

$$
\eta_{\mu\nu} V^\mu V^\nu = -m^2. \tag{1.22}
$$

Not only is this constant throughout spacetime, it is precisely equal to the norm of the momentum vector of a particle. Thus, this gradient has the straightforward interpretation as the momentum of this monochromatic wave.

In curved spacetime, the picture changes somewhat. Not only does the form of (1.17) become more complicated, the notion of lines and planes no longer make sense, much less the concept of a plane wave. It is therefore unclear what solutions to the Klein-Gordon equation will look like in this new spacetime. Manifolds, however, have the property that they locally "appear flat." More formally, at every event (excluding singularities) it is always possible to choose coordinates such that in a neighborhood of the event, $g_{\mu\nu} \approx \eta_{\mu\nu}$. Therefore, within a small enough region the solutions are more or less plane waves. Small is a rather imprecise notion. The size of this region depends ultimately on the wave solutions themselves. Intuitively, if a wave oscillates oscillates many times (in time or space) before the metric components vary too much within a region, the manifold will appear approximately flat to the wave, and so its deviations from the plane wave solution will be minimal. More concretely, this implies that $\partial_\sigma g_{\mu\nu} \ll k$. Thus, in the limit $k \to \infty$ the resulting solutions should behave locally as plane wave solutions.

This corresponds to the limit of geometric or ray optics and this limit should determine curves along which these waves travel. Intuitively, these should curves should be geodesics, but this remains to be seen and will be discussed further in Chapter 3.

1.2 Wavefunction of the Universe

In 1983, Stephen Hawking and James Hartle published a ground state solution of the Wheeler-DeWitt equation - an attempt at bridging the disconnect between quantum mechanics and GR [2]. This solution had the interesting feature of a "no-boundary" condition. Classical cosmological models describe a singularity in the past corresponding to the Big Bang. At this event, all matter in the universe was compressed into a single point. In the mathematics of GR, the volume element vanishes and the predictive power of the theory breaks down. Singularities are notoriously difficult to deal with in classical mechanics as infinities make no physical sense. Quantum mechanics, however, handles them just fine. As the authors put it:

> This is analogous to the behavior of the wave function of the electron in the hydrogen atom. In a classical treatment, the situation in which the electron is at the proton is singular. However, in a quantum-mechanical treatment the wave function in a state of zero angular momentum is finite and nonzero at the proton. This does not cause any problems in the case of the hydrogen atom. In the case of the Universe we would interpret the fact that the wave function can be finite and nonzero at the zero three-geometry as allowing the possibility of topological fluctuations of the three-geometry [2].

The singularity is physical, but it does not cause a breakdown in predictive power as it might classically. The topological fluctuation they talk about can refer to something relatively benign like a change from a spherical geometry to a flat one or a more radical change like a change in signature. A signature change would transform the manifold from Lorentzian in signature (one negative eigenvalue of the metric with the others being positive) to Riemannian (the metric is positive definite and all eigenvalues are positive). In this region, time would behave as a spatial coordinate and the physics should change radically. Hawking and Hartle say that this structure might eliminate some problems with the big bang model.

This thesis investigates a small part of this problem. I introduce a few spacetimes that exhibit this character and attempt to determine the behavior of test particles and fields in such a spacetime. At the transition between Riemannian and Lorentzian pieces, the physics become difficult to interpret and require a few more tools to fully characterize. Asymptotic methods can aid in this interpretation and the relevant topics are discussed in the following section.

1.3 Asymptotic Analysis

Equations of motion in GR tend to be complicated since the both the Einstein field equations and the geodesic equation are highly non-linear. Therefore approximations are often used to determine behavior of solutions in certain limits. Typically, this involves a series approximation to the solution, expanded in terms of some small parameter. Taylor series, for instance, are local expansions in that the small parameter is the distance from some center point x_0 to the position x where the value of the function is desired. This approximation, however useful, fails to capture the behavior at points far away from the center point since the distance is no longer "small." In this instance, far more terms need to be computed than the average physicist's attention span will allow. A different tool is required.

Global techniques attempt to determine solution behavior in some asymptotic limit over the entire, or large potion of the domain of interest. While local series approximations require many, sometimes hundreds, of terms to approximate points outside a small neighborhood, global expansions require only a few terms to glean interesting behavior about the system. The usual procedure is to find or introduce a small parameter and express the solution in terms of a power series in this parameter. The coefficients in this series may either be constants or functions over the domain of interest. Because these are asymptotic series, there is no guarantee that adding more terms will improve the result; the series may in fact diverge. The choice of parameter is often critical in determining this behavior

The Wentzel-Kramers-Brillouin (WKB) approximation is one such global technique. This method is applicable when the highest order derivative of an ordinary differential equation is multiplied by a small parameter ε. For instance, such a differential equation may be

$$\varepsilon y^{(m)}(x) + \sum_{i=0}^{m-1} p_i(x) y^{(i)}(x) = 0 \tag{1.23}$$

where $p_i(x)$ are functions (not necessarily analytic) over the domain of $y(x)$ that are "larger" than ε and $y^{(i)}(x)$ is the ith derivative of y. The WKB approximation assumes asymptotic solutions of the form

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right], \tag{1.24}$$

where δ is a small parameter and the $S_n(x)$ are non-constant, slowly varying functions. Substituting this relation into (1.23) gives, by dominant balance, a sequence of equations for the $S_n(x)$ functions. For this approximation to be valid, the series in the exponent of (1.24) needs to be a genuine asymptotic series; each term must be successively smaller than the last. Concretely, this means that

$$\delta^n S_{n+1}(x) \ll \delta^{n-1} S_n(x) \tag{1.25}$$

as $\delta \to 0, \forall n$ holds over the entire domain of interest. If the series is truncated at $n = N$, then this truncated series is a good approximation to the actual solution if the next term satisfies

$$\delta^N S_{N+1}(x) \ll 1 \tag{1.26}$$

as $\delta \to 0$ [3].

Chapter 2

Signature Changing Spacetimes

This Chapter sketches the properties of three universes that exhibit a transition of between Lorentzian and Riemannian regions of the manifold. These models are purposefully not realistic models of the universe, but are merely tools to get a feel for the behavior at the boundary. Properties like the curvature, geodesics, and Klein-Gordon solutions are determined and issues with the transition are discussed, including the need for asymptotic analysis.

2.1 $1+1$ Paraboloid

Many $1+d$ dimensional spacetimes are constructed by embedding a $1+d$ dimensional hypersurface in a $1+(d+1)$ dimensional Minkowski spacetime. For instance, de Sitter space can be defined by embedding a $1+d$ dimensional hyperboloid of a single sheet in Minkowski spacetime. Depending on the type of surface, this can result in Lorentzian manifolds, Riemannian manifolds, or manifolds that have both sets of behaviors. Embedding a paraboloid within $1+2$ Minkowski spacetime results in such a mixed character. Figure 2.1 shows such an embedding. Here, x and y are the two spatial coordinates of the Minkowski spacetime while σ is the timelike coordinate.

2.1.1 Properties

The surface has an explicit definition as $\sigma = x^2+y^2$, but this is inconvenient for discussing physics within the surface. Instead, parametrize the surface by $\sigma = t^2$, $x = t\cos\theta$, and $y = t\sin\theta$ for $t \in [0,\infty)$ and $\theta \in [0, 2\pi)$. Thus, the metric for such a spacetime can be given by

$$ds^2 = -\left(4t^2-1\right)dt^2 + t^2 d\theta^2. \tag{2.1}$$

In these coordinates, the metric becomes degenerate at $t = 1/2$ when the $g_{00} = -\left(4t^2-1\right)$ term vanishes. This is the boundary between the Riemannian ($t < 1/2$) and Lorentzian ($t > 1/2$) regions. It is necessary to characterize this spacetime to understand the effect of this boundary on both matter and radiation within the spacetime. While GR fails in $1+1$ dimensions[1] the curvature can still inform the dynamics of the situation.

Using $x^0 = t$ and $x^1 = \theta$, the non-zero Christoffel symbols of the paraboloid are

$$\Gamma^0_{00} = \frac{4t}{4t^2-1}$$
$$\Gamma^0_{11} = \frac{t}{4t^2-1}$$

[1] The Einstein tensor vanishes and therefore the connection between curvature and matter breaks down in this 1+1 dimensional universe. The stress energy tensor must vanish as well.

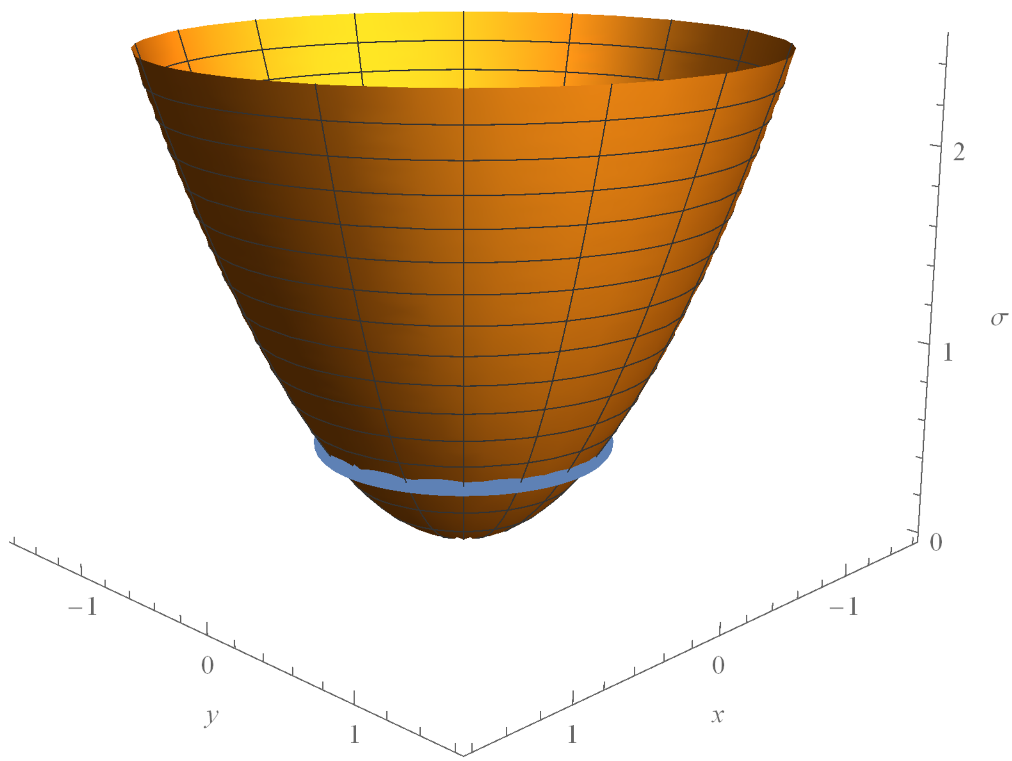

Figure 2.1: Embedding of the $1+1$ paraboloid in $1+2$ Minkowski spacetime. The blue curve denotes the location of the boundary.

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{t}$$

The only independent non-zero component of the Riemann tensor is

$$R^0{}_{101} = \frac{-4t^2}{\left(4t^2-1\right)^2}$$

Thus, the Ricci tensor components are

$$R_{00} = \frac{4}{4t^2-1} = \frac{-4}{(4t^2-1)^2} g_{00}$$

$$R_{11} = \frac{-4t^2}{(4t^2-1)^2} = \frac{-4}{(4t^2-1)^2} g_{11}$$

$$\Rightarrow R_{\mu\nu} = \frac{-4}{(4t^2-1)^2} g_{\mu\nu}$$

The Ricci Scalar is

$$R = \frac{-8}{(1-4t^2)^2}$$

Now the Einstein tensor vanishes ($G_{00} = G_{11} = 0$) as expected in a $1+1$ dimensional manifold. Many of the Christoffel symbols, the Riemann tensor, the Ricci tensor, and scalar curvature all diverge at the transition. In the higher dimensional analogues of this spacetime, this may lead to infinities in density or pressure if a similar divergence occurs. This is explored in Section (2.2).

2.1.2 Geodesics

Geodesics on this manifold satisfy

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{t}\frac{dt}{d\lambda}\frac{d\theta}{d\lambda} = f(\lambda)\frac{d\theta}{d\lambda} \tag{2.2}$$

$$\frac{d^2t}{d\lambda^2} + \frac{4t}{4t^2-1}\left(\frac{dt}{d\lambda}\right)^2 + \frac{t}{4t^2-1}\left(\frac{d\theta}{d\lambda}\right)^2 = f(\lambda)\frac{dt}{d\lambda} \tag{2.3}$$

These equations are not exactly solvable for even an affine parameter. If, however, coordinate time is used as the parameter, solutions can be found. In this case, for some affine parameter σ,

$$\begin{aligned} f(t) &= -\frac{d^2t}{d\sigma^2}\left(\frac{dt}{d\sigma}\right)^{-2} \\ &= \frac{4t}{4t^2-1} + \frac{t}{4t^2-1}\left(\frac{d\theta}{d\sigma}\right)^2\left(\frac{d\sigma}{dt}\right)^2 \\ f(t) &= \frac{4t}{4t^2-1} + \frac{t}{4t^2-1}\left(\frac{d\theta}{dt}\right)^2 \end{aligned} \tag{2.4}$$

where (2.3) for an affine parameter was inserted in the second line. Inserting this into (2.2), yields

$$\frac{d^2\theta}{dt^2} = \frac{2-4t^2}{t(4t^2-1)}\frac{d\theta}{dt} + \frac{t}{4t^2-1}\left(\frac{d\theta}{dt}\right)^3. \tag{2.5}$$

The trivial solution $d\theta/dt = 0$ exists, but this equation can be integrated to determine more involved $d\theta/dt$. Thus, I obtain curves where θ is constant or that satisfy

$$\frac{d\theta}{dt} = \begin{cases} \pm\frac{1}{t}\sqrt{\frac{1-4t^2}{Ct^2-1}} & t < \frac{1}{2} \\ \pm\frac{1}{t}\sqrt{\frac{4t^2-1}{1-Ct^2}} & t > \frac{1}{2} \end{cases} \tag{2.6}$$

In the Lorentzian region, $C < 0$ for time-like geodesics so that the argument under the radical remains real. This is easily seen by taking the norm of the tangent vector using this parametrization.

$$\begin{aligned} g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} &= -\left(4t^2-1\right)\left(\frac{dt}{dt}\right)^2 + t^2\left(\frac{d\theta}{dt}\right)^2 \\ g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} &= -\ Ct^2\left(\frac{4t^2-1}{Ct^2-1}\right) \end{aligned} \tag{2.7}$$

Geodesics along constant θ are recovered in the limit $C \to -\infty$. Changing variables in this expression to some affine parameter $t \to \sigma$ should force the norm to -1. By the chain rule,

$$\frac{dx^\mu}{dt} = \frac{d\sigma}{dt}\frac{dx^\mu}{d\sigma}, \tag{2.8}$$

and thus,

$$\left(\frac{d\sigma}{dt}\right)^2 = Ct^2\left(\frac{4t^2-1}{Ct^2-1}\right). \tag{2.9}$$

Integrating, this yields a rather messy expression for an affine parameter in terms of coordinate time.

$$\sigma(t)=\sqrt{\frac{(4t^2-1)(Ct^2-1)}{4C}}+\frac{4-C}{4C}\ln\left[C\sqrt{4t^2-1}+2\sqrt{C}\sqrt{Ct^2-1}\right]+\sigma_0 \tag{2.10}$$

This gives a sense for how the proper time of an object varies with coordinate time. This equation will find use in comparing these exact geodesic solutions to approximate ones discussed in Section 4.2.

It is easy to show that null geodesics satisfy $C=0$.

$$\frac{d\theta_{null}}{dt}=\pm\frac{\sqrt{4t^2-1}}{t} \tag{2.11}$$

This can be integrated to obtain an explicit equation for the null geodesics for $t>\frac{1}{2}$

$$\theta_{null}(t)=\pm\sqrt{4t^2-1}\mp\frac{1}{2}\arccos\left(\frac{1}{4t}\right)+\phi \tag{2.12}$$

where ϕ is an arbitrary phase. At the interface, this curve has the properties

$$\theta_{null}\left(\frac{1}{2}\right)=\mp\frac{\pi}{3}+\phi,\ \frac{d\theta_{null}}{dt}\left(\frac{1}{2}\right)=0. \tag{2.13}$$

While null geodesics do not exist in the Riemannian region of the manifold, any massless particle must match these conditions in the transition. Sensible solutions for $t<\frac{1}{2}$ also have the condition that $C<0$. Again this can be seen by taking the norm of the tangent vectors.

$$g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=Ct^2\left(\frac{1-4t^2}{Ct^2-1}\right) \tag{2.14}$$

This quantity should be positive for $t<1/2$, which only occurs for negative C. An interesting property is that every geodesic in the Riemannian region can match a boundary condition for a null geodesic in the Lorentzian region. Seemingly there is no way to distinguish between masses until they exit the Riemannian region, if such a statement has any meaning at all.

This is resolved through the use of asymptotic analysis. Performing a local expansion around the transition breaks this ambiguity, but this analysis is beyond the scope of this thesis.

2.1.3 Klein-Gordon Equation

I now briefly show the form that solutions to the Klein-Gordon equation take in this spacetime. Inserting the metric components into (1.17) yields

$$\frac{-t^2}{4t^2-1}\left[\ddot{\phi}-\frac{1}{t(4t^2-1)}\dot{\phi}+m^2\left(4t^2-1\right)\phi\right]+\phi''=0 \tag{2.15}$$

where dots and primes indicate differentiation by t and θ respectively. If solutions of the form $\phi(t,\theta)=T(t)R(\theta)$ are assumed, then this equation is separable into two ordinary differential equations.

$$\ddot{T}-\frac{1}{t(4t^2-1)}\dot{T}+\frac{4t^2-1}{t^2}\left(m^2t^2+k^2\right)=0 \tag{2.16}$$

$$R''+k^2R=0 \tag{2.17}$$

The second equation admits solutions of the form

$$R(\theta)=A\cos(k\theta)+B\sin(k\theta) \tag{2.18}$$

Since the metric is periodic in θ - the timelike curves defined by $\theta = 0$ and $\theta = 2\pi$ are identified - so are the wave solutions. Therefore $k \in \mathbb{Z}$. When $t > 1/2$ the time dependent equation has solutions

$$T(t) = C\cos\left(\xi(t)\right) + D\sin\left(\xi(t)\right) \tag{2.19}$$

where

$$\xi(t) = k\left[\sqrt{4t^2-1} + \arctan\left(\frac{1}{\sqrt{4t^2-1}}\right)\right]. \tag{2.20}$$

When $t < 1/2$ the time dependent solutions become

$$T(t) = Ce^{-\zeta(t)} + De^{\zeta(t)} \tag{2.21}$$

where

$$\zeta(t) = k\left[\sqrt{1-4t^2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{4t^2-1}}\right)\right]. \tag{2.22}$$

It is possible to then match the solutions in the two regions. The usual way of doing this runs into issues since the transition is a singularity in the Klein-Gordon equation. Again, asymptotic analysis techniques exist that define a unique way of doing this.

2.2 $1+3$ Paraboloid

In this many dimensions it is difficult to obtain solutions for either geodesics or the Klein-Gordon equation. My purpose here, then, is to merely outline further issues with signature changing spacetimes by showing some properties and their implications.

It is straightforward to generalize the paraboloid to higher dimensions. By embedding a paraboloid with the equation $\sigma = x^2 + y^2 + z^2 + w^2$ in $1+4$ Minkowski spacetime, the induced metric on this hypersurface is

$$ds^2 = (1-4t^2)dt^2 + t^2[d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi d\chi^2)]. \tag{2.23}$$

Setting $x^0 = t$, $x^1 = \theta$, $x^2 = \phi$, and $x^3 = \chi$, and grinding through the numerous intermediate steps (found in Appendix (A)) the components of the Einstein Tensor are

$$G_{00} = 12 \tag{2.24}$$

$$G_{11} = -4t^2\frac{4t^2-3}{(4t^2-1)^2} \tag{2.25}$$

$$G_{22} = -4t^2\frac{4t^2-3}{(4t^2-1)^2}\sin^2\theta \tag{2.26}$$

$$G_{33} = -4t^2\frac{4t^2-3}{(4t^2-1)^2}\sin^2\theta\sin^2\phi \tag{2.27}$$

Assume the stress-energy tensor takes the form of a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\, g_{\mu\nu} \tag{2.28}$$

where ρ is the energy density, p is the pressure, and U_μ is the velocity of the fluid. The U_i must vanish since the off-diagonal terms in the Einstein tensor vanish. Therefore $U_0 = \sqrt{4t^2-1}$ since $g^{\mu\nu}U_\mu U_\nu = -\left(4t^2-1\right)^{-1}\left(U_0\right)^2 = -1$. Thus

$$\rho = \frac{3}{2\pi G_N}\frac{1}{4t^2-1} \tag{2.29}$$

$$p = -\frac{1}{2\pi G_N}\frac{4t^2-3}{(4t^2-1)^2} = -\frac{\rho}{3}\frac{4t^2-3}{4t^2-1} \tag{2.30}$$

Note that both of these terms go to infinity at the boundary. The pressure is also negative for $t > \sqrt{3}/2$, making the perfect fluid assumption questionable, behaving instead more like a cosmological constant model.

2.3 Simple Metric

2.3.1 Properties

Both paraboloid metrics are daunting in their complexity. Both in the form of their geodesics and the matter content required to create them. A simpler model is desired that may be able to glean more information from this kind of spacetime. One can create such a simple spacetime by defining

$$ds^2 = -t\,dt^2 + h_{ij}dx^i dx^j \tag{2.31}$$

where h_{ij} is a d dimensional spatial metric dependent only on x^i of slices of constant coordinate time. With the coordinate transformation $d\sigma = \sqrt{t}dt$ this may appear to be the same as the well behaved metric

$$ds^2 = -d\sigma^2 + h_{ij}dx^i dx^j. \tag{2.32}$$

This metric is not so well behaved since, if $t \in (-\infty, \infty)$, then $\sigma \in (-i\infty, 0)\bigcap[0, \infty)$, which is characteristically different from Minkowski space. While very artificial, this spacetime clearly has a transition at $t = 0$ from a Riemannian space to a Lorentzian spacetime.

The only nonzero Christoffel symbols are

$$\Gamma^0_{00} = \frac{1}{2t} \tag{2.33}$$

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2}h^{il}\left(\partial_j h_{kl} + \partial_k h_{lj} - \partial_l h_{jk}\right). \tag{2.34}$$

The only non-zero component of the Riemann tensor is

$$R^i{}_{jkl} = r^i{}_{jkl} \tag{2.35}$$

where $r^i{}_{jkl}$ is the Riemann tensor on the spacelike slices. The time dependence has dropped out completely from the spatial curvature. It renters, however in the Einstein tensor since the metric enters directly.

$$G_{00} = \frac{t}{2}r \tag{2.36}$$

$$G_{ij} = r_{ij} - \frac{1}{2}r\,h_{ij} \tag{2.37}$$

The Einstein tensor is perfectly well behaved until it becomes degenerate at the transition. Assuming a perfect fluid stress-energy tensor again the density and pressure are

$$\rho = \frac{r}{16\pi G_N} \tag{2.38}$$

$$p = \frac{-(d-2)\,r}{16\pi G_N} = -(d-2)\rho \tag{2.39}$$

For $d = 3$, $p = -\rho$ which corresponds to a vacuum dominated universe. Again, this revelation is not terribly surprising considering the origins of this spacetime.

2.3.2 Geodesics

The geodesic equation for this manifold is

$$\frac{d^2t}{d\lambda^2} + \frac{1}{2t}\left(\frac{dt}{d\lambda}\right)^2 = f(\lambda)\frac{dt}{d\lambda} \tag{2.40}$$

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = f(\lambda)\frac{dx^i}{d\lambda} \tag{2.41}$$

Geodesics on the spacelike surfaces of constant coordinate time define the x^i components of the geodesics on the spacetime. If an affine parameter σ is used, (2.40) can be integrated to

$$t(\sigma) = C(\sigma - \sigma_0)^{2/3}. \tag{2.42}$$

where C and σ_0 are constants of integration. A simple coordinate transformation recovers the linear relation between t and σ.

2.3.3 Klein-Gordon Equation

The Klein-Gordon Equation for the simple metric is

$$-\frac{1}{t}\left[\ddot{\phi} - \frac{1}{2t}\dot{\phi}\right] + \left[h^{ij}\partial_i\partial_j\phi + \left(\frac{1}{2}\frac{\partial_i h}{h}h^{ij} + \partial_i h^{ij}\right)\partial_j\phi\right] = m^2\phi. \tag{2.43}$$

Defining $\phi(x^\mu) = T(t)R(x^i)$, this separates into

$$\frac{1}{t}\ddot{T} - \frac{1}{2t^2}\dot{T} + \left(k^2 + m^2\right)T = 0 \tag{2.44}$$

$$h^{ij}\partial_i\partial_j R + \left[\frac{1}{2}\frac{\partial_i h}{h}h^{ij} + \partial_i h^{ij}\right]\partial_j R + k^2 R = 0. \tag{2.45}$$

The spatial equation cannot be solved exactly until the h^{ij} are specified. The time dependent equation, however, admits solutions of the form

$$T(t) = A\cos\left(\frac{2}{3}\sqrt{k^2 + m^2}t^{3/2}\right) + B\sin\left(\frac{2}{3}\sqrt{k^2 + m^2}t^{3/2}\right). \tag{2.46}$$

In the Lorentzian region, the solution oscillates as expected. In the Riemannian region, however, the trigonometric functions turn into hyperbolic functions and can decay or grow exponentially.

Chapter 3

WKB Approximation

The goal of this Chapter is to determine if some link exists between the geometric optics paths that Klein-Gordon waves take in curved spacetime and the geodesics that particles follow. The answer to this question provides a method of finding approximate geodesics in not only signature changing spacetimes, but those that conform to the form of the generic metric introduced at the onset of the Chapter. There is ample reason to expect these waves to follow geodesics. Light follows null geodesics by definition and deviation from this behavior in the wave description would be problematic. Massive fields should be held to a similar standard and should follow timelike geodesics.

3.1 A Generic Spacetime

Consider a $1 + d$ dimensional manifold M of the form

$$ds^2 = -a\, dt^2 + b\, h_{ij}\, dx^i dx^j \tag{3.1}$$

a and b are functions of time and the h_{ij} are functions of spatial coordinates only. h_{ij} can be interpreted as the metric on spacelike submanifolds of this spacetime. Both b and the h_{ij} functions are positive definite, but a is allowed to switch sign. The metric is general enough that it can describe important spacetimes (such as the FLRW metric, Isotropic spacetimes, the Lemaître–Tolman metric under special conditions, and the paraboloid metric from Section 2.1), but has the important property of separability. This is critical to producing approximate solutions, since it lets us convert a linear partial differential equation of $1 + d$ variables to $1 + d$ linear ordinary differential equations which are in principle far easier to solve.

3.1.1 Properties

Before continuing into the investigation of waves in this spacetime, I will spend some time delving into the properties of this spacetime. For this manifold, the non-zero Christoffel coefficients are:

$$\Gamma^0_{00} = \frac{\dot{a}}{2a} \tag{3.2}$$

$$\Gamma^0_{ij} = \Gamma^0_{ji} = \frac{\dot{b}}{2a} h_{ij} \tag{3.3}$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{b}}{2b} \delta^i_j \tag{3.4}$$

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2} h^{il} \left(\partial_j h_{kl} + \partial_k h_{lj} - \partial_l h_{jk} \right). \tag{3.5}$$

From these, the geodesics can be determined. The curvature can also be calculated through the Riemann tensor. The independent non-zero components are:

$$R^i{}_{0j0} = -\frac{1}{2}\delta^i_j \left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{b}^2}{b^2} \right] \tag{3.6}$$

$$R^i{}_{jkl} = r^i{}_{jkl} + \frac{1}{2}\frac{b}{a}\frac{\dot{b}^2}{b^2}\delta^i_{[k}h_{l]j}. \tag{3.7}$$

Here, $r^i{}_{jkl}$ denotes the Riemann tensor on the spacelike submanifold. Contracting, I find the Ricci tensor and scalar:

$$R_{00} = -\frac{d}{2}\left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{b}^2}{b^2} \right] \tag{3.8}$$

$$R_{ij} = r_{ij} + \frac{1}{2}\frac{b}{a}h_{ij}\left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{d-2}{2}\frac{\dot{b}^2}{b^2} \right] \tag{3.9}$$

$$R = \frac{1}{b}\left\{ r + d\frac{b}{a}\left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{d-3}{4}\frac{\dot{b}^2}{b^2} \right] \right\} \tag{3.10}$$

Interestingly, the Ricci scalar separates between a spatial dependent part (scaled by the factor b) and a time dependent part. Finally, I construct the Einstein tensor:

$$G_{00} = \frac{1}{2}\frac{a}{b}r + \frac{d(d-1)}{8}\frac{\dot{b}^2}{b^2} \tag{3.11}$$

$$G_{ij} = r_{ij} - \frac{1}{2}r\,h_{ij} - \frac{d-1}{2}\frac{b}{a}h_{ij}\left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{d-4}{4}\frac{\dot{b}^2}{b^2} \right] \tag{3.12}$$

Note that these components vanish for $d = 1$, as they should, since the Einstein tensor is zero in $1+1$ dimensions. For $d \neq 1$ more information can be gleaned. As before, assuming the stress-energy tensor takes the form of a perfect fluid.

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\,g_{\mu\nu} \tag{3.13}$$

these components can give the matter content of the spacetimes they describe. The fluid must be at rest since the $G_{i0} \propto U_i U_0$ and $G_{i0} = G_{0j}$ vanish. Therefore $U_0 = \sqrt{a}$ since $g^{\mu\nu}U_\mu U_\nu = -1$ and

$$\rho = \frac{1}{16\pi G_N}\left[\frac{r}{b} + \frac{d(d-1)}{4a}\frac{\dot{b}^2}{b^2} \right] \tag{3.14}$$

$$p = \frac{-1}{16\pi G_N}\left\{ \frac{(d-2)\,r}{b} + \frac{d\,(d-1)}{a}\left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{d-4}{4}\frac{\dot{b}^2}{b^2} \right] \right\}. \tag{3.15}$$

Both density and pressure terms have the potential to misbehave as $a, b \to 0$. In fact, at the transition between the Riemannian and Lorentzian regions they both become infinite. Unlike in the paraboloid spacetime, this pressure term at least has a chance at being non-negative.

3.1.2 Geodesics

While the geodesic equation can be easily written down, solving for the geodesic curves in practice is a more difficult matter. The resulting coupled ordinary differential equations are highly non-linear and can only be solved once the specific function of the metric are known. Using (1.8) and the Christoffel symbols for this manifold, the geodesic equation states:

$$\frac{d^2x^0}{d\lambda^2} + \frac{\dot{a}}{2a}\left(\frac{dx^0}{d\lambda}\right)^2 + \frac{\dot{b}}{a}h_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = f(\lambda)\frac{dx^0}{d\lambda} \tag{3.16}$$

$$\frac{d^2x^i}{d\lambda^2} + \frac{\dot{b}}{b}\frac{dx^0}{d\lambda}\frac{dx^i}{d\lambda} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = f(\lambda)\frac{dx^i}{d\lambda} \tag{3.17}$$

It should be clear that these can not be solved directly except for very fortunate choices for the parametrization variable λ aided, perhaps, with the benefit of hindsight. Like all nonlinear equations, initial conditions are all important in the behavior of the solution and two very different expressions could result from different conditions. In addition, approximation techniques are notoriously ill-behaved for nonlinear equations since no one technique works for all types of equations. For these reasons, the geometric optics approach seems very appealing. Unlike most of the governing equations of relativity, the Klein-Gordon equation is linear and therefore far more predictable in behavior. Showing that the ray optics curves produced by the Klein-Gordon equation can produce geodesics, however, is another matter.

Both (3.16) and (3.17) are ordinary differential equations since the tangent vectors depend only on the location along the geodesic; they only vary with the parameter. In contrast, the velocity of waves will be a vector field over the entire manifold and will thus be dependent on $1 + d$ variables. In order to make the connection between the two, there has to be some link between vector fields and tangent vectors to curves. A technique from the study of differential equations - especially for those that have no analytical solution - is to represent the differential equation of interest as a vector field. Solutions - known as integral curves - are then everywhere tangent to this field. A similar method is used here; I construct curves that are everywhere tangent to some vector field V^μ, then determine the conditions this vector field must satisfy such that the curves are geodesics.

For a vector field V^μ, I define a curve $x^\mu(\lambda)$ such that it is everywhere tangent to V^μ. I will include a normalizing factor of m^{-1} designed[1] to make the norm of the tangent vectors equal to -1.

$$\frac{dx^\mu}{d\lambda} = \frac{1}{m}V^\mu\left[x^\nu(\lambda)\right] \tag{3.18}$$

The derivative of (3.18) with respect to λ is

$$\begin{aligned}\frac{d^2x^\mu}{d\lambda^2} &= \frac{1}{m}\frac{d}{d\lambda}V^\mu \\ &= \frac{1}{m}\frac{dx^\nu}{d\lambda}\partial_\nu V^\mu \\ &= \frac{1}{m^2}V^\nu\partial_\nu V^\mu \end{aligned} \tag{3.19}$$

Inserting (3.18) and (3.19) into (1.8), the geodesic equation becomes

$$m^{-2}V^\nu\partial_\nu V^\mu + m^{-2}\Gamma^\mu_{\nu\sigma}V^\nu V^\sigma = fm^{-1}V^\mu \tag{3.20}$$

Rearranging,

[1] The introduction of this factor is motivated by the considering the norm of the exact solution to the Klein-Gordon in Minkowski spacetime in (1.22). This will be seen to be a valid generalization later. A change of parameter or defining V^μ differently could accomplish the same task.

$$V^\nu \nabla_\nu V^\mu = m\, f\left(x^\alpha\right) V^\mu \tag{3.21}$$

Thus, for an arbitrary vector field to be able to define geodesics, $V^\nu \nabla_\nu V^\mu$ must either vanish or be parallel to V^μ, depending on the parametrization the vector field admits.

3.1.3 Klein-Gordon Equation

The Klein-Gordon equation, (1.17), is the primary object of study for this Chapter and is reproduced here:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = m^2\phi \tag{3.22}$$

For this manifold $-g = -ab^d h$ and $h = \det\left(h_{ij}\right)$. Applying the ansatz of separability - that is, $\phi\left(x^0, x^i\right) = T\left(x^0\right) R\left(x^i\right)$- the following relations are obtained for the time, and spatially dependent portions respectively:

$$\frac{b}{a}\ddot{T} + \frac{b}{a}\left[\frac{d}{2}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\right]\dot{T} + (bm^2 + k^2)T = 0 \tag{3.23}$$

$$h^{ij}\partial_i\partial_j R + \left[\frac{1}{2}\frac{\partial_i h}{h}h^{ij} + \partial_i h^{ij}\right]\partial_j R + k^2 R = 0 \tag{3.24}$$

In the traditional WKB approximation, the background field is taken to vary slowly. In quantum mechanical problem this field is the potential; in this case, it's the metric components. To reproduce geometric optics, the momentum is assumed to be very large. The integration constant k can be interpreted as the wavenumber of the wavefunction and, since $\hbar = 1$, the momentum of the wave. Thus, k is assumed to be large. In Section 1.3 the WKB approximation was stated to work when the leading derivative was multiplied by a small parameter. Dividing (3.23) and (3.24) by k^2 the validity of this tool for this situation becomes more certain.

$$\varepsilon\ddot{T} + \varepsilon\left[\frac{d}{2}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\right]\dot{T} + \left[a\varepsilon m^2 + \frac{a}{b}\right]T = 0 \tag{3.25}$$

$$\varepsilon h^{ij}\partial_i\partial_j R + \varepsilon\left[\frac{1}{2}\frac{\partial_i h}{h}h^{ij} + \partial_i h^{ij}\right]\partial_j R + R = 0 \tag{3.26}$$

where $\varepsilon = k^{-2}$. Since $k \to \infty$, the terms that are not differentiated dominate the differential equation. This appears to conform to the WKB approximation requirements. The functions T and R take the form

$$T = e^S = \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right] \tag{3.27}$$

$$R = e^Q = \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n Q_n\right] \tag{3.28}$$

where S is a function of time, and Q is a function of the spatial coordinates.

3.1.3.1 Time Dependent Terms

Inserting (3.27) into (3.23) and identifying $\delta = k^{-1}$ gives a differential equation that can be split into powers of k.

$$k^2\left[\frac{b}{a}\dot{S_0}^2+1\right]+k\left[\frac{b}{a}\ddot{S_0}+2\frac{b}{a}\dot{S_0}\dot{S_1}+\frac{b}{a}\left(\frac{d}{2}\frac{\dot{b}}{b}-\frac{1}{2}\frac{\dot{a}}{a}\right)\dot{S_0}\right]$$
$$+\left[\frac{b}{a}\ddot{S_1}+2\frac{b}{a}\dot{S_0}\dot{S_2}+\frac{b}{a}\dot{S_1}^2+\frac{b}{a}\left(\frac{d}{2}\frac{\dot{b}}{b}-\frac{1}{2}\frac{\dot{a}}{a}\right)\dot{S_1}+bm^2\right]+\mathcal{O}\left(\frac{1}{k}\right)=0 \quad (3.29)$$

The differential equation has been expanded out to order k^0 because this is first order where mass comes into play. Note that the leading order term has the form of the eikonal equation from geometric optics. In order to be asymptotically consistent for large k, each term is set to zero. The first three functions in the series for S are

$$S_0 = \pm i\int_{t_0}^{t}\sqrt{\frac{a}{b}}dt', \quad (3.30)$$

$$S_1 = -\frac{d-1}{4}\ln\left(\frac{b}{b_0}\right), \quad (3.31)$$

and

$$S_2 = \pm\frac{i}{2}\int_{t_0}^{t}\left\{m^2\sqrt{ab}-\mathrm{sgn}(a)\frac{d-1}{4}\sqrt{\frac{b}{a}}\left[\frac{\ddot{b}}{b}-\frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{d-3}{4}\frac{\dot{b}^2}{b^2}\right]\right\}dt', \quad (3.32)$$

Interestingly, the S_2 term can be expressed in terms of the scalar curvature as

$$S_2 = \pm\frac{i}{2}\int_{t_0}^{t}\sqrt{ab}\left\{m^2-\mathrm{sgn}(a)\frac{d-1}{4d}\left[R-\frac{r}{b}\right]\right\}dt'. \quad (3.33)$$

While the manifold has been written in a Lorentzian fashion, (3.30),(3.31), and (3.32) have been calculated to allow for sign changes in a. Simply substitute $a \to -a$ to obtain the appropriate functions for a Riemannian manifold. Note that, as expected, the Lorentzian solutions oscillate and the Riemannian solutions decay or grow exponentially in time.

3.1.3.2 Space Dependent Terms

Inserting (3.28) into (3.24) gives a similar differential equation to the time dependent relation.

$$k^2\left[h^{ij}\partial_i Q_0\partial_j Q_0+1\right]+k\left[h^{ij}\partial_i\partial_j Q_0+2h^{ij}\partial_i Q_0\partial_j Q_1+\left(\frac{1}{2}\frac{\partial_i h}{h}h^{ij}+\partial_i h^{ij}\right)\partial_j Q_0\right]$$
$$+\left[h^{ij}\partial_i\partial_j Q_1+2h^{ij}\partial_i Q_0\partial_j Q_2+h^{ij}\partial_i Q_1\partial_j Q_1+\left(\frac{1}{2}\frac{\partial_i h}{h}h^{ij}+\partial_i h^{ij}\right)\partial_j Q_1\right]+\mathcal{O}\left(\frac{1}{k}\right)=0 \quad (3.34)$$

Notice that, again, the eikonal equation appears in the highest order term. Not much can be done to glean information from this equation without further specifying the form that the metric takes. The lowest order function Q_0 is impossible to obtain without losing generality. In principle, once the h_{ij} are specified the Q_n are easily determined. As will be seen in Section 3.3, I can still obtain suggestive results with the gradient of Q. If a $1+1$ dimensional manifold is assumed, however, then the Q_n can be found to arbitrary order.

3.2 1+1 Solution

3.2.1 General Properties

For a 1+1 dimensional manifold, the time dependent functions S_n follow from (3.30), (3.31), and (3.32) and are

$$S_0 = \pm i \int_{t_0}^{t} \sqrt{\frac{a}{b}} dt', \tag{3.35}$$

$$S_1 = 0, \tag{3.36}$$

and

$$S_2 = \pm i \frac{m^2}{2} \int_{t_0}^{t} \sqrt{ab} dt'. \tag{3.37}$$

Thus, in the Lorentzian region, the time dependent piece is

$$T(t) = \alpha \cos \xi(t) + \beta \sin \xi(t) \tag{3.38}$$

where

$$\xi(t) = k \int_{t_0}^{t} \left(\sqrt{\frac{a}{b}} + \frac{1}{2} \left(\frac{m}{k} \right)^2 \sqrt{ab} \right) dt'. \tag{3.39}$$

The spatial differential equation (3.34) becomes

$$k^2 \left[\frac{1}{h} Q_0'^2 + 1 \right] + k \left[\frac{1}{h} Q_0'' + \frac{2}{h} Q_0' Q_1' - \frac{1}{2} \frac{h'}{h^2} Q_0' \right] + \left[\frac{1}{h} Q_1'' + \frac{2}{h} Q_0' Q_2' + \frac{1}{h} Q_1'^2 - \frac{1}{2} \frac{h'}{h^2} Q_1' \right] + \mathcal{O}\left(\frac{1}{k} \right) = 0. \tag{3.40}$$

The functions in the series, then, are

$$Q_0 = \pm i \int_{x_0}^{x} \sqrt{h} dx' \tag{3.41}$$

$$Q_1 = constant \tag{3.42}$$

$$Q_2 = constant. \tag{3.43}$$

Therefore, the space-dependent solution for the 1+1 Lorentzian manifold is

$$R(x) = \gamma \cos \theta(x) + \delta \sin \theta(x) \tag{3.44}$$

where

$$\theta(x) = k \int_{x_0}^{x} \sqrt{h} dx'. \tag{3.45}$$

The full WKB approximate solution to the 1+1 Klein-Gordon equation (in the Lorentzian region) is

$$\phi\left(x^{\mu}\right) = A \cos(\theta(x) - \xi(t) + \eta_+) + B \cos(\theta(x) + \xi(t) + \eta_-) \tag{3.46}$$

A, B, and $\eta_\pm$ are constants. In the Riemannian region, the sign of a flips and S becomes entirely real; Q is unaffected. Therefore, the solution is

$$\phi\left(x^\mu\right) = Ae^{-\xi}\cos\left(\theta(x) + \eta_+\right) + Be^{\xi}\cos\left(\theta(x) + \eta_-\right). \tag{3.47}$$

An important property that falls out of the approximation is that, in the Lorentzian region at least, the leading order terms in the series expansions for S and Q are purely imaginary. This is critical to the validity of the WKB approximation. If S_0 had a real part, for example, then the amplitude of ϕ would grow or decay significantly over a wavelength or period. Built into the approximation is the requirement that the locally the spacetime looks like Minkowski spacetime, and therefore the solutions should appear like plane waves. The limit $k \to \infty$ ensures that variations in the manifold are miniscule compared with the wavelength or period. Thus, the amplitude should remain more or less constant as well.

The geometric optics limit usually assumes this behavior, as in an argument by Percacci [4]. If for instance, the solution was assumed to take the form $\phi = Ae^{iS}$ where A and S are real valued functions of the coordinate functions, then (1.16) takes the form

$$\nabla^\mu\nabla_\mu A - A\nabla^\mu S\nabla_\mu S + i\left(2\nabla^\mu S\nabla_\mu A + A\nabla^\mu\nabla_\mu S\right) = m^2 A. \tag{3.48}$$

Now, if you assume that the amplitude A changes much more slowly than S and m^2 ($\nabla_\mu S \gg \nabla_\mu A, \nabla_\mu S \gg m^2$) then it follows that $\nabla_\mu S\nabla^\mu S \approx 0$. In our notation, $V^\mu V_\mu \approx 0$ which means that V_μ can be used to define null geodesics. The approximation developed in this thesis is, in the end, equivalent to Percacci's.

3.2.2 Velocity

Consider the WKB solution in the Lorentzian region. Without loss of generality, assume $\eta_\pm = 0$ and consider only at the "right" moving wave $(B = 0)$[2]. I define a "velocity" of the wave as described in Section 1.1.3 by taking the gradient of the phase.

$$V^\mu = g^{\mu\nu}\partial_\nu\left(\theta(x) - \xi(t)\right) \tag{3.49}$$

For simplicity, the lowered index is used for now. In the geometric optics limit, only the leading order term of the WKB approximation is used. Therefore, this "velocity" is, in components,

$$V^0 = \frac{k}{\sqrt{ab}}, \tag{3.50}$$

$$V^1 = \frac{k}{b\sqrt{h}}. \tag{3.51}$$

The norm of this gradient will indicate whether the field is timelike, spacelike, or null. The norm is

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu &= -a\left(V^0\right)^2 + bh\left(V^1\right)^2 && (3.52)\\ &= -\frac{k^2}{b} + \frac{k^2}{b} && (3.53)\\ &= 0 && (3.54)\end{aligned}$$

Therefore the vector field is null in the geometric optics limit and can therefore be used to define null geodesics. This analysis does not include the effect of mass on the vector field. To consider this effect, terms up to S_2 and Q_2 must be included. The "velocity" of the massive field, therefore is

[2]Including both the "left" and "right moving waves not only complicates the gradient and the following analysis, it also doesn't allow for satisfying interpretation. It creates cross terms in the inner product that can be removed by averaging but, perhaps more importantly, it doesn't allow for a clean analogy to particle mechanics. Classically, a particle will not simultaneously move left and right and it makers no sense to consider the norm of a velocity constructed with the two directions.

$$V^0 = \frac{k}{\sqrt{ab}}\left[1+\frac{b}{2}\left(\frac{m}{k}\right)^2\right], \tag{3.55}$$

$$V^1 = \frac{k}{b\sqrt{h}}, \tag{3.56}$$

and the norm of this is given by

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu &= -a\left(V^0\right)^2 + bh\left(V^1\right)^2 \\ &= -a\left[\frac{k^2}{ab}+\frac{m^4}{4k^2}\frac{b}{a}+\frac{m^2}{a}\right]+\frac{k^2}{b}. \end{aligned}$$

Thus, to highest order[3],

$$g^{\mu\nu}V_\mu V_\nu = -m^2 \tag{3.57}$$

This shows that the tangent vectors are timelike or null depending on the value of m. The "velocity" is that of light only if $m = 0$. Interestingly, this inner product is exactly the norm of momenta of particles, suggesting a more concrete interpretation of V_μ as a wave momentum. The velocity, then would be given by $m^{-1}V_\mu$.

In Riemannian regions of the manifold this definition fails since S flips from pure imaginary to real and thus $V_0 = 0$.

3.2.3 Connection to the Geodesic Equation

For convenience, the Christoffel coefficients are reproduced here.

$$\Gamma^0_{00} = \frac{\dot{a}}{2a} \tag{3.58}$$

$$\Gamma^0_{ij} = \Gamma^0_{ji} = \frac{\dot{b}}{2a}h_{ij} \tag{3.59}$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{b}}{2b}\delta^i_j \tag{3.60}$$

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2}h^{il}\left(\partial_j h_{kl} + \partial_k h_{lj} - \partial_l h_{jk}\right) \tag{3.61}$$

Using these values, $V^\nu \nabla_\nu V^\mu$ can be calculated. To order k^0 - the k^{-1} terms are non-existent[4] - the terms on the left hand side of (3.21) are

$$V^\nu \partial_\nu V^0 = -\frac{1}{2a}\left[\frac{k^2}{b}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)+m^2\frac{\dot{a}}{a}\right] \tag{3.62}$$

$$V^\nu \partial_\nu V^1 = -\frac{k^2}{b}\left[\frac{1}{\sqrt{abh}}\frac{\dot{b}}{b}+\frac{1}{2bh}\frac{h'}{h}\right]-\frac{m^2}{2\sqrt{abh}}\frac{\dot{b}}{b} \tag{3.63}$$

[3] The k^{-2} term is ignored since the S_3 and Q_3 functions will also produce terms of this order.

[4] This is because both S_1 and $S_3 \in \mathbb{R}$ and the only contributions to order k^{-1} in $V^\nu \nabla_\nu V^\mu$ would involve products of these terms. Since only imaginary terms contribute to the phase, they are ignored. Terms on the order of k^{-2} arise, but these are ignored since S_4 terms contribute.

$$V^\nu \Gamma^0_{\nu\sigma} V^\sigma = \frac{1}{2a}\left[\frac{k^2}{b}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)+m^2\frac{\dot{a}}{a}\right] \tag{3.64}$$

$$V^\nu \Gamma^1_{\nu\sigma} V^\sigma = \frac{k^2}{b}\left[\frac{1}{\sqrt{abh}}\frac{\dot{b}}{b}+\frac{1}{2bh}\frac{h'}{h}\right]+\frac{m^2}{2\sqrt{abh}}\frac{\dot{b}}{b} \tag{3.65}$$

Therefore, since $V^\nu \nabla_\nu V^\mu = V^\nu \partial_\nu V^\mu + V^\nu \Gamma^\mu_{\nu\sigma} V^\sigma$,

$$V^\nu \nabla_\nu V^0 = V^\nu \nabla_\nu V^1 = 0 \tag{3.66}$$

Evidently the condition $f(\lambda) = 0$ is required for this vector field. The integral curves then, are affinely parametrized geodesics. Geodesics, for some affine parameter λ, are described by a system of parametric ordinary differential equations:

$$\frac{dx^0}{d\lambda} = \frac{k}{m}\frac{1}{\sqrt{a\left[x^0(\lambda)\right]b\left[x^0(\lambda)\right]}}\left[1+\frac{b\left[x^0(\lambda)\right]}{2}\left(\frac{m}{k}\right)^2\right] \tag{3.67}$$

$$\frac{dx^1}{d\lambda} = \frac{k}{m}\frac{1}{b\left[x^0(\lambda)\right]\sqrt{h\left[x^1(\lambda)\right]}} \tag{3.68}$$

The explicit functional dependence of the tangent vectors is shown to give an idea of the seemingly insurmountable task of calculating geodesics from this result. It seems that I have simply replaced one exact nonlinear differential equation for another, even more nonlinear equation. The new equations are better, however, in the sense that they are only first order rather than second order, simplifying numerical integration and enabling analytical solutions in some cases.

3.3 $1 + d$ Solution

3.3.1 General Properties and Velocity

While (3.34) permits no explicit solutions, we can still explore the implicit properties of the solution. Looking at only the highest order terms, a general property can be stated. (3.30) is already applicable to an $1 + d$ dimensional spacetime. The equivalent statement for the space-dependent expansion is

$$h^{ij}\partial_i Q_0 \partial_j Q_0 = -1 \tag{3.69}$$

This cannot be directly solved for Q_0 without specifying more about the metric. It will, however, be useful in determining the properties of the velocity vectors. It is reasonably clear that the Q_0 must be complex. This is because $\left[h^{ij}\right]$ (a symmetric real valued (0-2)-rank tensor field) can be written through an appropriate coordinate change as a diagonal matrix $\left[h^{i'j'}\right] = \text{diag}\left(h^{0'0'}, h^{1'1'}, ..., h^{d'd'}\right)$. In this form, the terms in the series consist of a product of the metric components (assumed to be positive definite) and the square of the partial derivatives of Q_0. Since the sum totals to a negative number, at least one of $\partial_i Q_0$ must be complex, giving solutions that oscillate and exponentially decay or grow in some directions. Since only real fields are of interest, we define the Klein-Gordon field as the sum of left and right moving waves.

$$\phi\left(x^\mu\right) = Ae^{k(Q_0-S_0+\eta_+)}+\bar{A}e^{k\left(\bar{Q}_0-\bar{S}_0+\bar{\eta}_+\right)}+Be^{k(Q_0+S_0+\eta_-)}+\bar{B}e^{k\left(\bar{Q}_0+\bar{S}_0+\bar{\eta}_-\right)} \tag{3.70}$$

where $A, B, \eta_\pm \in \mathbb{C}$ are constants. Without loss of generality, let $\eta_\pm = 0$ and consider only the "right" moving wave ($B = 0$). In considering the velocity of the wave, care must be taken to only consider the imaginary part of both Q_0 and S_0. S_0 is already known to be purely imaginary, but Q_0 could be complex. I define a gradient vector in the usual way, but with a lowered index[5] as

$$V_\mu = \partial_\mu \left(k\frac{Q_0 - \bar{Q}_0}{2i} - k\int_{t_0}^{t} \sqrt{\frac{a}{b}} dt' \right). \tag{3.71}$$

The first term in the argument selects only the imaginary part of Q_0. In the Lorentzian region, the components of the gradient are

$$V_0 = -k\sqrt{\frac{a}{b}},$$

$$V_i = \frac{k}{2i}\left(\partial_i Q_0 - \partial_i \bar{Q}_0\right).$$

Taking the norm yields

$$\begin{aligned} g^{\mu\nu}V_\mu V_\nu &= -\frac{1}{a}V_0^2 + \frac{1}{b}h^{ij}V_iV_j \\ &= -\frac{k^2}{b} - \frac{k^2}{4b}h^{ij}\left(\partial_i Q_0\partial_j Q_0 + \partial_i\bar{Q}_0\partial_j\bar{Q}_0 - \partial_i Q_0\partial_j\bar{Q}_0 - \partial_i\bar{Q}_0\partial_j Q_0\right) \end{aligned}$$

From (3.69) and its complex conjugate, the first two terms in the parentheses are equal to -1. What is left, then, is

$$g^{\mu\nu}V_\mu V_\nu = -\frac{k^2}{2b}\left[1 - h^{ij}\partial_i Q_0 \partial_j \bar{Q}_0\right] \tag{3.72}$$

Let $Q_0 = q_{re} + iq_{im}$ where $q_{re}, q_{im} : M_d \to \mathbb{R}$. Then

$$h^{ij}\partial_i Q_0\partial_j Q_0 = \left(h^{ij}\partial_i q_{re}\partial_j q_{re} - h^{ij}\partial_i q_{im}\partial_j q_{im}\right) + i\left(h^{ij}\partial_i q_{re}\partial_j q_{im}\right) \tag{3.73}$$

$$h^{ij}\partial_i Q_0\partial_j \bar{Q}_0 = h^{ij}\partial_i q_{re}\partial_j q_{re} + h^{ij}\partial_i q_{im}\partial_j q_{im} \tag{3.74}$$

The real part of (3.73) equals -1 by (3.69), while the imaginary part must vanish. Inserting this result into (3.74), (3.72) becomes

$$g^{\mu\nu}V_\mu V_\nu = \frac{k^2}{b}h^{ij}\partial_i q_{re}\partial_j q_{re} \tag{3.75}$$

This expression is ≥ 0 for any real-valued analytic function q_{re}. Since mass does not enter until the third term in the WKB expansion, this quantity should equal zero. As discussed above, I do not expect the leading order functions to have a real part since the amplitude would grow or decay significantly over a wavelength or period. If this is not convincing, Appendix B attempts to argue this further.

3.3.2 Connection to the Geodesic Equation

Like Section 3.2.3, the game here is to determine whether the velocity field is everywhere tangent to geodesics by determining whether 3.21 holds. Although I don't expect there to be a real part, this analysis retains that possibility for completeness. The relevant vector field components are given by

$$V^0 = \frac{k}{\sqrt{ab}} \tag{3.76}$$

[5]This is purely to simplify the notation.

$$V^i = \frac{k}{2bi} h^{ij} \partial_j \left(Q_0 - \bar{Q}_0\right) \tag{3.77}$$

Thus,

$$V^\nu \partial_\nu V^0 = -\frac{k^2}{2ab} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) \tag{3.78}$$

$$V^\nu \partial_\nu V^i = -\frac{k^2}{\sqrt{ab^3}} \frac{\dot{b}}{b} \frac{1}{2i} h^{ij} \partial_j \left(Q_0 - \bar{Q}_0\right) - \frac{k^2}{4b^2} h^{mn} \partial_n \left(Q_0 - \bar{Q}_0\right) \partial_m \left[h^{ij} \partial_j \left(Q_0 - \bar{Q}_0\right)\right] \tag{3.79}$$

$$V^\nu \Gamma^0_{\nu\sigma} V^\sigma = \frac{k^2}{2ab} \left[\frac{\dot{a}}{a} + \frac{1}{2} \frac{\dot{b}}{b} \left(1 + h^{nm} \partial_m Q_0 \partial_n \bar{Q}_0\right)\right] \tag{3.80}$$

$$V^\nu \Gamma^i_{\nu\sigma} V^\sigma = \frac{k^2}{\sqrt{ab^3}} \frac{\dot{b}}{b} \frac{1}{2i} h^{ij} \partial_j \left(Q_0 - \bar{Q}_0\right) - \frac{k^2}{4b^2} \Gamma^i_{nm} h^{nj} h^{ml} \partial_j \left(Q_0 - \bar{Q}_0\right) \partial_l \left(Q_0 - \bar{Q}_0\right) \tag{3.81}$$

Finally, after some index gymnastics, these reduce to

$$V^\nu \nabla_\nu V^0 = -\frac{k^2}{4ab} \frac{\dot{b}}{b} \left(1 - h^{nm} \partial_m Q_0 \partial_n \bar{Q}_0\right) \tag{3.82}$$

and

$$V^\nu \nabla_\nu V^i = -\frac{k^2}{4b^2} h^{ij} \partial_j \left(h^{nm} \partial_m Q_0 \partial_n \bar{Q}_0\right) \tag{3.83}$$

These components vanish if the criteria for null vectors ($h^{ij} \partial_i Q_0 \partial_j \bar{Q}_0 = 1$) is met. Additionally, it is easy to show that these equations reduce to those of the $1+1$ dimensional spacetime. If these quantities did not vanish, it is difficult to imagine what function Q_0 would allow $V^\nu \nabla_\nu V^\mu$ and V^μ to be parallel.

3.4 Limitations of the WKB Approximation

Section 1.3 gives a set of criteria that must be satisfied in order for the WKB approximation to be effectively used. They are, in our notation

$$\frac{1}{k^n} S_{n+1} \ll \frac{1}{k^{n-1}} S_n \tag{3.84}$$

$$\frac{1}{k^N} S_{N+1} \ll 1 \tag{3.85}$$

as $k \to \infty$, $\forall t$, x^i where N is the last term in the truncated WKB series. The first term implies

$$\lim_{k \to \infty} \left| \frac{S_{n+1}}{k S_n} \right| = 0 \tag{3.86}$$

while the second requires

$$\lim_{k \to \infty} \left| \frac{1}{k^N} S_{N+1} \right| = 0. \tag{3.87}$$

In the $1+d$ case, it is difficult to determine the validity of the spatial dependent series, but the utility of the time dependent series can be shown. (3.87) says that truncating the series after a single term is invalid since

$$\lim_{k\to\infty}\left|\frac{d-1}{4}\ln\left(\frac{b}{b_0}\right)\right| \neq 0 \tag{3.88}$$

except in the cases where $d = 1$ or b is constant. After two terms, however, the approximation approaches validity. (3.86) and (3.87) respectively imply

$$\lim_{k\to\infty}\left|\frac{d-1}{4k}\frac{\ln(b/b_0)}{\int_{t_0}^{t}\sqrt{a/b}dt'}\right| = 0 \tag{3.89}$$

and

$$\lim_{k\to\infty}\left|\frac{1}{2k}\int_{t_0}^{t}\left\{m^2\sqrt{ab} - \mathrm{sgn}(a)\frac{d-1}{4}\sqrt{\frac{b}{a}}\left[\frac{\ddot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{d-3}{4}\frac{\dot{b}^2}{b^2}\right]\right\}dt'\right| = 0. \tag{3.90}$$

Clearly these limits do converge to zero , except for in the limit as $a, b \to 0$ or $a, b \to \infty$ as it would near a transition between Riemannian and Lorentzian region in spacetime or near a metric singularity. This is expected since the metric can no longer be seen as slowly varying in this limit, and the plane wave picture breaks down. In principle, you could approximate solutions arbitrarily close to a singularity simply by increasing k until as close as required. A boundary layer of invalid solutions will always exist between the singularity and regions of spacetime where the approximation is well behaved.

In the case where $d = 1$, the solution behaves in a way that is not expected. The solutions no longer have this boundary layer when $a, b \to 0$ and the approximation appears perfectly convergent. Even when truncating after three terms, the solutions do not lose this character, since S_1 is an arbitrary constant,

$$\lim_{k\to\infty}\left|\frac{1}{2kS_1}\int_{t_0}^{t} m^2\sqrt{ab}dt'\right| = 0, \tag{3.91}$$

and, because $S_3 = \frac{m^2}{4}\int_{t_0}^{t} a\frac{\dot{b}}{b}dt'$,

$$\lim_{k\to\infty}\left|\frac{1}{k^2}S_3\right| = \lim_{k\to\infty}\left|\frac{m^2}{4k^2}\int_{t_0}^{t} a\frac{\dot{b}}{b}dt'\right| = 0. \tag{3.92}$$

As will be seen in the following chapter, the WKB approximation converges to the exact solutions, evidently even up to the transition. The reason for this is not known, but warrants further investigation.

Chapter 4

Applications of the WKB Approximation

This Chapter applies the approximate solutions to specific manifolds to test for their validity by; examining the degree to which they approximate the exact solution to the Klein-Gordon equation, and determining whether the geodesics obtained in the $1+1$ case approximate the exact ones, where known.

4.1 Minkowski Spacetime

In 1+1 Minkowski Spacetime $a = b = h = 1$. Solving the Klein-Gordon equation directly yields the following relation:

$$\phi = A\cos\left[\sqrt{k^2+m^2}t - kx + \eta_+\right] + B\sin\left[\sqrt{k^2+m^2}t - kx + \eta_+\right] \tag{4.1}$$

In the approximation scheme, $\xi(t) = k\left(1+\frac{1}{2}\left(\frac{m}{k}\right)^2\right)(t-t_0)$, $\theta = k(x-x_0)$, and the WKB solution takes the form

$$\phi(x^\mu) = A\cos\left[kx - k\left(1+\frac{1}{2}\left(\frac{m}{k}\right)^2\right)t + \eta_+\right] + B\sin\left[kx + k\left(1+\frac{1}{2}\left(\frac{m}{k}\right)^2\right)t + \eta_-\right] \tag{4.2}$$

In the large k limit, these two equations are equivalent, since the term under the square root approximates as $\sqrt{k^2+m^2} = k\sqrt{1+\left(\frac{m}{k}\right)^2} \approx k\left(1+\frac{1}{2}\left(\frac{m}{k}\right)^2\right)$. When $m = 0$, this approximation becomes exact. Using (3.67) and (3.68), I can compute approximate geodesics. These equations become

$$\frac{dt}{d\tau} = \frac{k}{m}\left[1+\frac{1}{2}\left(\frac{m}{k}\right)^2\right] \tag{4.3}$$

and

$$\frac{dx}{d\tau} = \frac{k}{m} \tag{4.4}$$

Integrating, I obtain parametric equations for straight lines.

$$t(\tau) = \frac{k}{m}\left[1+\frac{1}{2}\left(\frac{m}{k}\right)^2\right](\tau-\tau_0) \sim \frac{\sqrt{k^2+m^2}}{m}(\tau-\tau_0) \tag{4.5}$$

$$x(\tau) = \frac{k}{m}(\tau-\tau_0) \tag{4.6}$$

These equations converge to the expected results. $\sqrt{k^2+m^2}/m$ is simply the Lorentz factor while k/m is the three velocity multiplied by the Lorentz factor since

$$v = \frac{p}{E} = \frac{k}{\sqrt{k^2 + m^2}} \tag{4.7}$$

4.2 Paraboloid

For the 1+1 paraboloid spacetime with metric

$$ds^2 = -\left(4t^2 - 1\right) dt^2 + t^2 d\theta^2 \tag{4.8}$$

the exact wave solution for $m = 0$ is

$$\phi\left(x^\mu\right) = A\cos\left[k\theta - k\left(\sqrt{4t^2-1} + \arctan\left(\frac{1}{\sqrt{4t^2-1}}\right)\right) + \eta_+\right] \\ + B\sin\left[k\theta + k\left(\sqrt{4t^2-1} + \arctan\left(\frac{1}{\sqrt{4t^2-1}}\right)\right) + \eta_-\right] \tag{4.9}$$

where $k \in \mathbb{Z}$. The WKB solution for arbitrary m is

$$\phi\left(x^\mu\right) = A\cos\left[k\theta - \xi - \eta_+\right] + B\sin\left[k\theta + \xi + \eta_-\right]$$

where

$$\xi(t) = k\left\{\sqrt{4t^2-1} + \arctan\left(\frac{1}{\sqrt{4t^2-1}}\right) + \frac{1}{24}\left(\frac{m}{k}\right)^2\left(4t^2-1\right)\right\} \tag{4.10}$$

Amazingly, the solutions match exactly for $m = 0$. This seems to be a general property of 1+1 Lorentzian manifolds, but doesn't extend to higher dimensions. The approximate geodesics are again given by (3.67) and (3.68) which, for this manifold and some affine parameter σ, become

$$\frac{dt}{d\sigma} = \frac{k}{m}\frac{1}{t\sqrt{4t^2-1}}\left[1 + \frac{t^2}{2}\left(\frac{m}{k}\right)^2\right] \tag{4.11}$$

$$\frac{d\theta}{d\sigma} = \frac{k}{m}\frac{1}{t^2}. \tag{4.12}$$

Combining the two conditions gives

$$\frac{d\theta}{dt} = \frac{\sqrt{4t^2-1}}{t}\left[1 + \frac{t^2}{2}\left(\frac{m}{k}\right)^2\right]^{-1}. \tag{4.13}$$

A comparison with the exact geodesic solutions (2.6) reveals a suggestive connection.

$$\sqrt{1 - Ct^2} \sim 1 + \frac{t^2}{2}\left(\frac{m}{k}\right)^2 \tag{4.14}$$

It would seem that $C = -\left(m/k\right)^2$, but without further terms this relationship is impossible to corroborate. Still, given that null geodesics are defined by $C = 0$ (automatically satisfied by $m = 0$) and particles at rest by $C \to \infty$ (again, satisfied by particles with zero momenta or $k \to 0$) this association seems appealing.

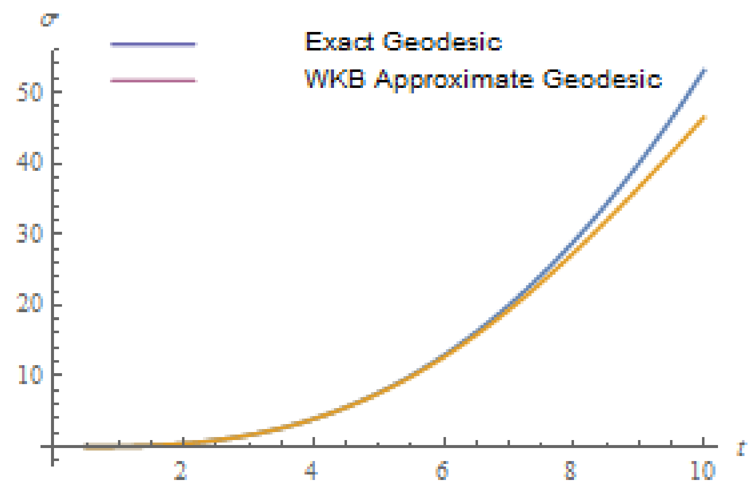

Figure 4.1: Affine parameter variation with $C = -(m/k)^2 = -0.01$ and $\sigma(t = 1/2) = 0$. The exact and approximate solutions closely match near where the transition where they are forced to match, but diverge away from this point.

This is further supported by examining the variation of proper time along a curve with respect to coordinate time. Integrating (4.11) gives an expression analogous to (2.10).

$$\begin{aligned}\frac{k}{m}(\sigma - \sigma_0) &= \int_{t_0}^{t} t\sqrt{4t^2-1}\left[1 + \frac{t^2}{2}\left(\frac{m}{k}\right)^2\right]^{-1} dt \\ &\approx \int_{t_0}^{t} t\sqrt{4t^2-1}\left[1 - \frac{t^2}{2}\left(\frac{m}{k}\right)^2\right] dt \\ \frac{k}{m}(\sigma - \sigma_0) &\approx \left(4t^2-1\right)^{3/2}\left[\frac{1}{12} - \frac{1}{240}\left(6t^2+1\right)\left(\frac{m}{k}\right)^2\right] \qquad (4.15)\end{aligned}$$

At first glance, this bears no resemblance to (2.10), but Figure 4.1 shows that the WKB approximate geodesics and exact geodesics follow one another closely if $C = -(m/k)^2$. This is by no means a rigorous demonstration of their equality, but hopefully will point the way to similar results for other spacetimes.

4.3 Simple Metric

The simpler signature changing spacetime in $1+1$ dimensions has a metric

$$ds^2 = -t\,dt^2 + dx^2. \qquad (4.16)$$

The WKB approximation gives

$$\phi(x^\mu) = A\cos\left[kx - k\frac{2}{3}t^{3/2}\left(1 + \frac{1}{2}\left(\frac{m}{k}\right)^2\right) + \eta_+\right] + B\cos\left[kx + k\frac{2}{3}t^{3/2}\left(1 + \frac{1}{2}\left(\frac{m}{k}\right)^2\right) + \eta_-\right] \qquad (4.17)$$

while the exact solution is simply

$$\phi(x^\mu) = A\cos\left[kx - \frac{2}{3}\sqrt{k^2+m^2}t^{3/2} + \eta_+\right] + B\cos\left[kx + \frac{2}{3}\sqrt{k^2+m^2}t^{3/2} + \eta_-\right]. \qquad (4.18)$$

Again, there appears to be a convergence to the exact solution when $m = 0$ as well as a fairly good approximation when $m \neq 0$. On this manifold, the approximate geodesics can be calculated from

$$\frac{dt}{d\sigma} = \frac{k}{m}\frac{1}{\sqrt{t}}\left[1 + \frac{1}{2}\left(\frac{m}{k}\right)^2\right] \tag{4.19}$$

and

$$\frac{dx}{d\sigma} = \frac{k}{m}. \tag{4.20}$$

Integrating yields

$$t(\sigma) = \frac{3}{2}\left[\frac{k}{m}\left[1 + \frac{1}{2}\left(\frac{m}{k}\right)^2\right](\sigma - \sigma_0)\right]^{2/3} \tag{4.21}$$

and

$$x(\sigma) = \frac{k}{m}(\sigma - \sigma_0) \tag{4.22}$$

Comparing to (2.42), it is easy to identify $C = \frac{3}{2}\left(\frac{k}{m}\right)^{2/3}\left(1 + \frac{1}{2}\left(\frac{m}{k}\right)^2\right)^{2/3}$. As in the paraboloid case, the integration constants can be related, albeit in a somewhat ugly way, to the ratio k/m.

4.4 FLRW Metric

The Friedman-Lemaître-Robertson-Walker (FLRW) metric is a proposed cosmological model for our universe. It assumes a homogeneous, isotropic universe that could either be globally flat or of some non-zero uniform curvature. The metric takes the form

$$ds^2 = -dt^2 + \alpha^2(t)d\Sigma^2 \tag{4.23}$$

where a is a function of time known as the scale factor, Σ are a set of coordinate functions on the spatial slices. This part of the metric can be written concisely as

$$d\Sigma = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \tag{4.24}$$

where K is the Gaussian curvature of the space when $\alpha(t) = 1$, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on S^2. I can now identify components in the WKB generic spacetime with this metric.

$$a = 0,\ b = \alpha^2$$

$$h_{rr} = \frac{1}{1 - Kr^2},\ h_{\theta\theta} = r^2,\ h_{\phi\phi} = r^2 \sin^2\theta$$

Thus, the WKB solution is

$$\phi = \frac{A}{\alpha}\cos\left[\theta\left(x^i\right) - \xi\left(t\right)\right] \tag{4.25}$$

where

$$\xi(t) = \int_{t0}^{t} \frac{k}{\alpha}\left\{1 + \alpha^2\left(\frac{m}{k}\right)^2 - \frac{\alpha\ddot{\alpha} + \dot{\alpha}^2}{k^2}\right\} dt' \tag{4.26}$$

and θ depends on the curvature of the spacetime as well as a choice of boundary conditions. If the curvature is zero and the spatial metric is Euclidean, then the problem reduces to Laplace's equation with an extra condition imposed upon it. The time dependent part shows that the frequency of oscillation gets smaller as the scale factor α grows. This corresponds to the universal redshift observed in receding objects.

Conclusions

This thesis explored the link between waves and geodesics in the large momentum ($k \to \infty$) limit. I showed that, to highest order, appropriately defined wave velocities point along null geodesics and can, in principle, be used to construct those curves. Beyond geometric optics, I showed that the velocities of massive solutions to the Klein-Gordon equation point along time-like geodesics. These approximations were shown to be valid in the regime where the solutions can be approximated as a plane wave, but break down for $d > 1$ at singularities in the metric. If k is allowed to be arbitrarily large, however, in principle you could approximate as close to rapidly varying regions of spacetime as desired.

Development of this tool stemmed from the desire to understand universes with Hawking-Hartle transitions between Riemannian and Lorentzian regions. Since the equations of motion become understandably ill-behaved at the transition, the machinery of General Relativity fails and new approaches are required. Especially since geodesics in these spacetimes are so complex, geometric optics seemed appealing as a way to grasp the physics of the situation. Coupled with local analysis of the singularities, this WKB approach could shed new light on the problem.

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Appendix A

Properties of the $1+3$ Paraboloid

The non-zero Christoffel coefficients are

$$\Gamma^0_{00} = \frac{4t}{4t^2-1}$$

$$\Gamma^0_{11} = \frac{t}{4t^2-1}$$

$$\Gamma^0_{22} = \frac{t\sin^2\theta}{4t^2-1}$$

$$\Gamma^0_{33} = \frac{t\sin^2\theta\sin^2\phi}{4t^2-1}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{t}$$

$$\Gamma^1_{22} = -\sin\theta\cos\theta$$

$$\Gamma^1_{33} = -\sin\theta\cos\theta\sin^2\phi$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \cot\theta$$

$$\Gamma^2_{33} = -\sin\phi\cos\phi$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\phi$$

A tedious calculation gives the independent non-zero components of the Riemann Tensor

$$R_{0101} = \frac{4t^2}{4t^2-1}$$

$$R_{0202} = \frac{4t^2\sin^2\theta}{4t^2-1}$$

$$R_{0303} = \frac{4t^2\sin^2\theta\sin^2\phi}{4t^2-1}$$

$$R_{1212} = \frac{4t^4\sin^2\theta}{4t^2-1}$$

$$R_{1313} = \frac{4t^4\sin^2\theta\sin^2\phi}{4t^2-1}$$

$$R_{2323} == \frac{4t^4\sin^4\theta\sin^2\phi}{4t^2-1}$$

Contracting yields the Ricci Tensor and scalar.

$$R_{00} = \frac{12}{4t^2 - 1}$$

$$R_{11} = 4t^2 \frac{8t^2 - 3}{(4t^2 - 1)^2}$$

$$R_{22} = 4t^2 \frac{8t^2 - 3}{(4t^2 - 1)^2} \sin^2 \theta$$

$$R_{33} = 4t^2 \frac{8t^2 - 3}{(4t^2 - 1)^2} \sin^2 \theta \sin^2 \phi$$

$$R = 48 \frac{2t^2 - 1}{(4t^2 - 1)^2}$$

Appendix B

Arguments for Vanishing Real Part of Q_0

B.1 Locally Inertial Coordinates

In this transformation, coordinates are chosen such that the spacetime metric evaluated at an event in the manifold equals the Minkowski metric ($g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$) and first derivatives of the metric vanish. For the spatial metric $h_{\hat{i}\hat{j}}$ in this coordinate system, the components are identically that of Euclidean space at this event. Now, rewriting (3.73) in a more familiar form for these coordinates:

$$\sum_{j=1}^{d}\left(\frac{\partial q_{re}}{\partial x^j}\right)^2 - \sum_{j=1}^{d}\left(\frac{\partial q_{im}}{\partial x^j}\right)^2 = -1 \tag{B.1}$$

$$\sum_{j=1}^{d}\frac{\partial q_{re}}{\partial x^j}\frac{\partial q_{im}}{\partial x^j} = 0 \tag{B.2}$$

These equations form a set of coupled first order nonlinear differential equations that can't be solved in general. A few properties, however, can be firmly established. The first is that the norm of the gradient of q_{im} is exactly 1 more than that of q_{re}. In addition, the gradients must be perpendicular. This severely restricts the space of solutions for this equation. The wave equation in Minkowski spacetime gives a clue as to the form of these solutions; the solution is separable in each coordinate, and the contributions are multiplicative. Thus, the argument of the exponential should be additive. Assume that

$$q_{re} = \sum_{j=1}^{d} q_{re,j}\left(x^j\right) \tag{B.3}$$

$$q_{im} = \sum_{j=1}^{d} q_{im,j}\left(x^j\right) \tag{B.4}$$

The functions q_{re} and q_{im} are thus decomposed into a linear combination of functions of a single variable. Now the partial derivatives in (B.1) and (B.2) become total derivatives of these new functions. Writing (B.1) now in a more suggestive form

$$\sum_{j=1}^{d}\left[\left(\frac{dq_{re,j}}{dx^j}\right)^2 - \left(\frac{dq_{im,j}}{dx^j}\right)^2\right] = -1 \tag{B.5}$$

Each term in the series is independent of the others, thus must equal a constant.

$$\left(\frac{dq_{re,j}}{dx^j}\right)^2 - \left(\frac{dq_{im,j}}{dx^j}\right)^2 = -n_j \tag{B.6}$$

$$\sum_{j=1}^{d} n_j = 1 \tag{B.7}$$

Similarly, for (B.2):

$$\frac{dq_{re,j}}{dx^j}\frac{dq_{im,j}}{dx^j} = \ell_j \tag{B.8}$$

$$\sum_{j=1}^{d} \ell_j = 0 \tag{B.9}$$

Now there are $2d + 2$ equations for $2d$ functions and $2d$ arbitrary constants. Using, (B.6) and (B.8) I can solve for the functions $q_{re,j}$ and $q_{im,j}$. From (B.6):

$$\frac{dq_{im,j}}{dx^j} = \pm\sqrt{\left(\frac{dq_{re,j}}{dx^j}\right)^2 + n_j} \tag{B.10}$$

and from (B.8):

$$\ell_j = \pm\frac{dq_{re,j}}{dx^j}\sqrt{\left(\frac{dq_{re,j}}{dx^j}\right)^2 + n_j} \tag{B.11}$$

Rearranging and using the quadratic formula yields

$$\left(\frac{dq_{re,j}}{dx^j}\right)^2 = \frac{1}{2}\left(-n_j + \sqrt{n_j^2 + 4\ell_j^2}\right) \equiv L_j^2 \tag{B.12}$$

Only the + solution makes physical sense, so the other has been dropped. Plugging this back into (B.6) yields

$$\left(\frac{dq_{im,j}}{dx^j}\right)^2 = \frac{1}{2}\left(n_j + \sqrt{n_j^2 + 4\ell_j^2}\right) \equiv \hat{k}_j^2 \tag{B.13}$$

Summing over the coordinates is equivalent to taking the norm of the gradients of q_{re} and q_{im}.

$$h^{\hat{i}\hat{j}}\partial_{\hat{i}} q_{re}\partial_{\hat{i}} q_{re} = \sum_{j=1}^{d} L_j^2 = L^2 \tag{B.14}$$

$$h^{\hat{i}\hat{j}}\partial_{\hat{i}} q_{im}\partial_{\hat{i}} q_{im} = \sum_{j=1}^{d} \hat{k}_j^2 = L^2 + 1 \tag{B.15}$$

where L is some constant. Lengths of vectors are conserved with coordinate changes, so

$$h^{ij}\partial_i q_{re}\partial_j q_{re} = L^2 \tag{B.16}$$

$$h^{ij}\partial_i q_{im}\partial_j q_{im} = L^2 + 1 \tag{B.17}$$

Thus, if L^2 can be forced to zero, the null condition can be recovered. In locally inertial coordinates, (B.12) and (B.13) imply

$$q_{re} = L_{\hat{j}} x^{\hat{j}} + C \tag{B.18}$$

$$q_{im} = \hat{k}_{\hat{j}} x^{\hat{j}} + D \tag{B.19}$$

The spatial component of the Klein-Gordon solution, then, becomes

$$R\left(x^j\right) \approx A \exp\left(k L_{\hat{j}} x^{\hat{j}} + i k_{\hat{j}} x^{\hat{j}}\right) \tag{B.20}$$

where $k_j = k\hat{k}_j$. The real term in the exponential causes the wave solution to blow up nonphysically at infinity. Therefore, $L_j = 0 \;\forall j$ and, happily, I obtain $L = 0$ and

$$g^{\mu\nu} V_\mu V_\nu = 0 \tag{B.21}$$

I have cheated in this last step of this argument; locally inertial coordinates are only valid in a neighborhood of an event in spacetime, yet I made an argument involving behavior at infinity. In Minkowski spacetime this argument is perfectly valid, however, and this line of reasoning is, at the very least suggestive of what must occur in curved spacetime. This analysis may not be entirely rigorous - or even convincing - but it at least points the way to why, physically, q_{re} must vanish.

B.2 WKB reformulation

In a different approximation scheme, the series expansion is done with two real functions - say A and S - one serving as the wave amplitude, the other is the argument. For the time dependent component of the wave T, rewrite it as

$$T(t) = A(t) e^{iS(t)}, \tag{B.22}$$

which leads to a new version of (3.23). The real and imaginary components of this are:

$$\frac{b}{a}\left[\ddot{A} - A\dot{S}^2\right] + \frac{b}{a}\left[\frac{d}{2}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\right]\dot{A} + \left(bm^2 + k^2\right) A = 0 \tag{B.23}$$

and

$$2\dot{A}\dot{S} + A\ddot{S} + \left(\frac{d}{2}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\right) A\dot{S} = 0 \tag{B.24}$$

Now, expand A and S in powers of k as before.

$$A(t) = \sum_{n=0}^{\infty} \frac{1}{k^n} A_n(t) \tag{B.25}$$

$$S(t) = k \sum_{n=0}^{\infty} \frac{1}{k^n} S_n(t) \tag{B.26}$$

Equations (B.23) and (B.24) are now, respectively:

$$k^2\left[-\frac{b}{a} A_0 \dot{S}_0^2 + A_0\right] + k\left[-\frac{b}{a} A_1 \dot{S}_0^2 - 2\frac{b}{a} A_0 \dot{S}_0 \dot{S}_1 + A_1\right] + \mathcal{O}\left(k^0\right) = 0 \tag{B.27}$$

$$k\left[2\dot{A}_0\dot{S}_0 + A_0\ddot{S}_0 + \left(\frac{d}{2}\frac{\dot{b}}{b} - \frac{1}{2}\frac{\dot{a}}{a}\right)A_0\dot{S}_0\right] + \mathcal{O}\left(k^0\right) = 0 \tag{B.28}$$

Looking at the two highest orders:

$$S_0 = \pm\int_{t_0}^{t}\sqrt{\frac{a}{b}}dt' \tag{B.29}$$

$$A_0 = \frac{A_{0i}}{b^{(d-1)/4}} \tag{B.30}$$

$$S_1 = constant \tag{B.31}$$

Comparing to (3.30) and (3.31), this replicates the previous solution for Lorentzian manifolds exactly. For the spatial part of the wave solution, I assume the form

$$R\left(x^i\right) = B\left(x^i\right)e^{iQ\left(x^i\right)} \tag{B.32}$$

where B and Q are real valued functions. Direct substitution into (3.24) leads to

$$h^{ij}\partial_i\partial_jB - Bh^{ij}\partial_iQ\partial_jQ + \left(\frac{1}{2}\frac{\partial_ih}{h}h^{ij} + \partial_ih^{ij}\right)\partial_jB + k^2B = 0 \tag{B.33}$$

$$2h^{ij}\partial_iB\partial_jQ + Bh^{ij}\partial_i\partial_jQ + \left(\frac{1}{2}\frac{\partial_ih}{h}h^{ij} + \partial_ih^{ij}\right)B\partial_jQ = 0 \tag{B.34}$$

Now expanding in powers of k:

$$B\left(x^i\right) = \sum_{n=0}^{\infty}\frac{1}{k^n}B_n\left(x^i\right) \tag{B.35}$$

$$Q\left(x^i\right) = k\sum_{n=0}^{\infty}\frac{1}{k^n}Q_n\left(x^i\right) \tag{B.36}$$

yields

$$k^2\left[-B_0h^{ij}\partial_iQ_0\partial_jQ_0 + B_0\right] + k\left[-B_1h^{ij}\partial_iQ_0\partial_jQ_0 - 2B_0h^{ij}\partial_iQ_0\partial_jQ_1 + B_1\right] + \mathcal{O}\left(k^0\right) = 0 \tag{B.37}$$

$$k\left[2h^{ij}\partial_iB_0\partial_jQ_0 + B_0h^{ij}\partial_i\partial_jQ_0 + \left(\frac{1}{2}\frac{\partial_ih}{h}h^{ij} + \partial_ih^{ij}\right)B_0\partial_jQ_0\right] + \mathcal{O}\left(k^0\right) = 0 \tag{B.38}$$

Solving this to highest order gives the desired

$$h^{ij}\partial_iQ_0\partial_jQ_0 = 1 \tag{B.39}$$

Appendix C

Alternate Velocity Definition

In the above treatment, I have defined the velocity of the Klein-Gordon solutions to be the gradient of their phase. In quantum mechanics, however, the gradient of the wavefunction itself is used. The Klein-Gordon equation as presented does not strictly define quantum fields since solutions violate unitarity, among other problems, but defining velocity in this manner can potentially shed some light on what happens in the Riemannian region since the phase has no “time” dependence. Let the velocity be defined by

$$V^{\mu} = g^{\mu\nu}\partial_{\nu}\phi. \tag{C.1}$$

For the Lorentzian region of the $1+1$ dimensional manifold, the components of this are

$$V^0 = -\frac{Ak}{\sqrt{ab}}\sin(\theta-\xi)\left[1+\frac{b}{2}\left(\frac{m}{k}\right)^2\right] \tag{C.2}$$

$$V^1 = -\frac{Ak}{b\sqrt{h}}\sin(\theta-\xi)\,. \tag{C.3}$$

To see if these can define geodesics, the components of $V^{\nu}\nabla_{\nu}V^{\mu}$ are

$$V^{\nu}\nabla_{\nu}V^0 = -\frac{A^2m^2k}{\sqrt{ab}}\sin(\theta-\xi)\cos(\theta-\xi) \tag{C.4}$$

$$V^{\nu}\nabla_{\nu}V^1 = -\frac{A^2m^2k}{b\sqrt{h}}\sin(\theta-\xi)\cos(\theta-\xi)\,. \tag{C.5}$$

Unlike the earlier definition, $V^{\nu}\nabla_{\nu}V^{\mu}$ does not vanish. Instead, it is proportional to V^{μ}, with the proportionality function

$$f(x^{\alpha}) = Am\cos(\theta-\xi)\,. \tag{C.6}$$

The Riemannian region is still difficult to handle, but not impossible. Given appropriate boundary conditions, this definition could be used to define a velocity. What this means, however, is unclear since in some sense this is a classically forbidden region. This is an area for further analysis.